4

Poincaré's Mathematical Creations in Search of the 'True Relations of Things'

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Can science teach us the true relations of things?

Henri Poincaré, *The Value of Science*, 1905.

4.1 INTRODUCTION

'Mathematical physics and pure analysis . . . mutually interpenetrate and their spirit is the same.'[1] Addressing (in absentia) the first International Congress of Mathematicians (ICM) in Zürich in 1897, Henri Poincaré (1854–1912)—French mathematician, mathematical physicist and philosopher—explained his vision of the mutual needs and shared spirit of these worlds, a perspective shaped by his deep conceptual work at that interface for almost two decades. 'Mathematics have a triple aim', he argued. It is 'not enough' that mathematics aims to 'furnish an instrument for the study of nature' so that the physicist could 'know it better', an aim notably exemplified by advances 'already rough-hewn' in celestial mechanics and mathematical physics. '[Mathematics must also] have a philosophical aim, and, I dare maintain, an aesthetic aim. They must aid the philosopher [and physicist] to fathom the notions of number, of space, of time.'[2]

Pure analysis—a battlefield long engaging the profound subtleties of infinitesimals and infinity, differentiability and continuity, number and the mathematical continuum—helped secure those advances in understanding nature (notably, Poincaré's new methods of celestial mechanics), but it also thereby deepened the needs and promises within mathematical physics. Mathematics, he asserted, must probe these depths with the 'aesthetic purpose' of enabling philosophers and physicists to grasp, articulate and expose such subtleties and relations within nature:

> The physicist cannot ask of the analyst to reveal to him a new truth; the [analyst] could at most only aid [the physicist] to foresee it. . . . All laws are therefore deduced from experiment; but to enunciate them, a special language is needful; ordinary language is too poor, it is besides too vague to express relations so delicate, so rich, and so precise. This therefore is

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> one reason why the physicist cannot do without mathematics: it furnishes him the only language he can speak.[3]

Poincaré's mathematical corpus attested to the richness of conceptual advance, pure and applied, that is possible within the intertwined fields of analysis and physics. Some among his audience would soon come to understand aspects of his vision and help advance it. Poincaré was just beginning another surge in activity at that interface, creating an array of utterly new mathematics (and associated philosophical discourse) that would ground his response to the radical new challenges within fin-de-siècle physics, a crisis within theory that greatly increased the stakes regarding our notions of number, space, time and, therewith, the physicists' ether.

In 1900, three international Congresses met in Paris—the second ICM, the first International Congress of Physicists (ICP) and the first International Congress of Philosophy (ICPHIL)—pushing the boundaries of open questions in the respective disciplines and their interconnections. In mathematics, emphasis on a fully rigorous axiomatic, strictly deductive, formal approach to proof advanced in tandem with a drive for a mathematics anchored in the certainties of 'number' within arithmetic rather than in subtleties (not yet deductively proven) about 'dimension' and the 'mathematical continuum' within analysis. In theoretical physics, experimental discoveries and electrodynamic theories challenged Newtonian notions of space, time and matter. Philosophical discourse about the meaning, methods and purpose of science stumbled on questions about space, absolute motion and the nature of the ether.

This chapter assesses the following question: how did Poincaré's vast corpus of mathematical innovation engage the rationale, and impact the fate, of the notion of the ether in physics? It asks what *Poincaré* was thinking, and it seeks understanding through his voice (speaking mathematics and philosophy), not through the arguments of contemporaries and later interpreters who did not grasp his full meaning (often completely distorting it), nor through categories foreign to his unique way of thinking. It finds that Poincaré had no ownership of the physicists' ether concept, and that he viewed the ether as neither necessary nor necessarily a hindrance for further advance. Rather, Poincaré attended to the profound and subtle needs within physics by creating profound and subtle mathematics—utterly new theoretical and interpretive concepts, tools and structures—to capture the 'true relations of things', rendering the physicists' ether superfluous to that goal while also creating mathematical structures for gravitational and quantum phenomena.

In his scientific practice and philosophy of science, Poincaré sought the 'true relations' that adhere in the phenomena—relations that persist irrespective of the choice of a metric geometry and a change in physical theory. This chapter is structured to aid understanding of how Poincaré's lifeworks 'hang together'[4]—how they cohere within Poincaré's way of thinking—which enables us to assess how his work instantiates what he means by the 'true relations of things' that unify physics.

Taking a historicist perspective anchored in detailed assessments of Poincaré's corpus and legacy by mathematicians and historians of mathematics, the chapter traces key strands in Poincaré's 1880s engagement with the subtleties of space and time and the structure we know as spacetime, and, concurrently, with the conceptual possibilities that opened to him as he sought to master and exploit topological intuitions in creating *analysis situs* (algebraic topology). It traces how Poincaré embedded these utterly new geometric and topological ways of thinking at the heart of pure mathematics, mathematical physics and philosophy.

Section 4.1 explains how Poincaré emphasised the need to break free from the geometry habituated by our senses, altered the discourse about the geometry of physical space and set out to create a new mathematics for relations 'so delicate, so rich, and so precise'. Section 4.2 examines Poincaré's 1891 essay, detailing how he maps the path from his 1880 hyperboloid model to his 1887 'fourth geometry'—offering different lenses into the geometry that would become Minkowski spacetime. Section 4.3 traces through Poincaré's philosophical writings (from 1901 to 1905) and his 1904 St. Louis address on the dynamics of the electron, documenting his view of the physicists' ether as a disposable 'garment' in the search for the 'true relations' that persist within natural phenomena. Section 4.4 examines the reaction to Poincaré's work at Göttingen, where Poincaré's corpus was assiduously studied and built upon, illustrating how Poincaré's geometric creations from 1880 to his four-dimensional geometric interpretation of the Lorentz transformation was a ready resource for Hermann Minkowski as he developed his spacetime geometry based on Einstein's theory of special relativity. Section 4.5 offers concluding comments about Poincaré's last year of life and his legacy, witnessing a juxtaposition of his works on space and time, analysis situs and dimension, quantum theory and statistical mechanics, and documenting the profundity that Poincaré and Einstein recognised in each other's works.

4.2 SEEKING A MATHEMATICS TO EXPRESS 'RELATIONS SO DELICATE, SO RICH, AND SO PRECISE'

The question of the geometry of physical space was rooted in early nineteenth-century discoveries regarding non-Euclidean geometries, intrinsic curvature and topological shape. Mathematicians mid-century were intrigued by topological possibilities for our space, but philosophers and physicists focused more on the question whether it might exhibit a positive or negative curvature at an astronomically large scale—whether ours might be a spherical world (with positive curvature) or a hyperbolic world (with negative curvature) rather than a Euclidean one (with zero curvature). The notion that one might empirically measure the curvature of our space by means such as stellar parallax became a tantalising possibility, indeed, an increasingly confident and explicit goal within informed scientific communities in the 1890s.

In an 1891 essay entitled '*Les géométries non euclidiennes*', Poincaré radically altered the possibilities and stakes regarding the question of the 'true' geometry and topology of our space and how we can come to know it.[5] In a highly provocative thought experiment, Poincaré explained how intelligent creatures from a hyperbolic world, whose geometry is based on its freely chosen non-Euclidean conventions, would, if transported to our world, observe the same phenomena we do but express the physical laws differently. We, likewise, would easily enunciate the laws of their world using our Euclidean conventions. Yet, in each such world, while understanding of the 'true relations' of its physical phenomena would be secure, no experiment would be able to determine *by metric means alone* the actual geometry of that world.

Poincaré's 'fiction' of the hyperbolic world (introduced in his 'Letter to Mouret')[6] challenged the validity of the assumptions of *geometric empiricism*—the claim that experiment (measurement) was sufficient to determine the geometry of cosmological space—while at the same time offering an alternative epistemology (*geometric conventionalism*) and promising a new mathematics, a new type of geometric reasoning (which he called 'analysis situs') that would help us 'find a way' to secure knowledge of the geometry and topology of spaces of higher dimension.[7]

As Poincaré explained in 1901, deep challenges throughout his varied research pushed him to engage in this ongoing mathematical quest and encouraged his trust in profound rewards for mathematics and physics: 'As for me, all the diverse paths on which I was successively engaged [1879–91] have led me to *Analysis Situs*.'[8] Poincaré recognised that mathematics, physics and celestial mechanics needed the certainty of theorems accessible only with this new type of reasoning.

Poincaré had entered into a philosophical discourse, couched in the language of neo-Kantian philosophy, which sought foundations for the geometry of our space. The discourse first centred on Hermann von Helmholtz's notion of freely moving rigid (measuring) bodies, then on infinitesimal transformation groups, becoming known as the Helmholtz–Lie (classical) space problem.[9] Poincaré saw the need to escape its metrical (measurement) requirements, and his philosophical stance was inextricably tied to his 1887 establishment of the 'fourth geometry' (explained in Section 4.3) and to the new mathematics he was creating.[10]

Poincaré's 'Analysis Situs' requires that our imagination break free from the geometry habituated by our senses, ignore metrical properties of geometric objects (properties involving measurement of distance and angles) and pursue the difficult art of 'reasoning well based on badly drawn figures' by focusing on relations unchanged by any continuous deformation.[11] The years 1887–91 mark the momentous interval during which Poincaré set himself the task to establish this new field of mathematics for the study of space. By 1895, Poincaré had single-handedly created the field called 'algebraic topology', developing over the next decade entirely new concepts, tools and intuitions to assess topological spaces of three and higher dimensions.[12]

The mathematical concept of the *amorphous continuum* provides the basis (realm of action) for mathematicians to rigorously conceptualise in higher dimensions, learning how we can 'supplement' our senses so as to reason in 'hyperspace'. The distinction between the mathematical (amorphous) continuum and the physical continuum of experience is crucial, as is Poincaré's emphasis that we do not have intuition about space itself. We hear Poincaré's philosophical voice, speaking mathematics, in the section entitled

'Qualitative Geometry' in his 1903 article 'Space and Its Three Dimensions',[13] which was reprinted as Chapter 3, 'The Notion of Space', in his 1907 collection of essays, *The Value of Science*:

> Euclidean space is not a form imposed upon our sensibility, since we can imagine non-Euclidean space; but the two spaces, Euclidean and non-Euclidean, have a common basis, that amorphous continuum of which I spoke in the beginning. From this continuum we can get either Euclidean space or Lobachevskian space . . . This continuum has a certain number of properties, *exempt from all idea of measurement* . . . The theorems of analysis situs have, therefore, this peculiarity that *they would remain true if the figures were copied by an inexpert draftsman* who should grossly change all the proportions and replace the straights by lines more or less sinuous. In mathematical terms, they are not altered by any 'point-transformation' whatsoever . . . Of all the theorems of analysis situs, the most important is that which is expressed in saying that space has three dimensions.[14]

Key to understanding subtleties of this need to pass from the world of the amorphous continuum to the world of space and geometry is Poincaré's claim that experience alone cannot 'engender mathematical notions', in particular the notion of mathematical continuity itself. Adding a *metric structure* to the amorphous continuum yields a *space* with attendant cosmological and epistemological implications.

In his works in analysis situs, Poincaré was after a precise mathematical notion of *dimensionality* anchored in the subtle conceptual tools and theorems he was creating. As he explained in 1908 to the fifth ICM in Rome, such deep penetration into unexplored terrains of thought can 'enable us really to see into hyperspace and to supplement our senses'.[15] Poincaré insisted that analysis situs is 'the only true domain of geometric intuition' and, once accessed, promised entry into vast new realms of mathematical activity.[16]

4.3 POINCARÉ'S 1891 ESSAY, FROM HIS 1880 HYPERBOLOID MODEL TO HIS 1887 FOURTH GEOMETRY: GEOMETRIC REASONING UNTETHERED, SPACE 'STRUCTURES' ENGAGED

Poincaré's essay '*Les géométries non euclidiennes*' appeared 15 December 1891 in a recently launched French journal with a diverse scientific audience for whom Poincaré sought to capture the *mathematical* and *epistemological* challenges confronting a geometric (*metrically empirical*) understanding of space.[17] Many scholars who assess the geometric conventionalism Poincaré introduced there focus on its conclusion, namely that, since hyperbolic and Euclidean metrics are inter-translatable, one can choose the simplest, Euclidean geometry. But Poincaré takes us to that conclusion by making a much deeper argument about geometry and what it yet lacks.

In a breathless litany, he takes us through Riemann's spherical world (finite without boundary) and the accelerating mathematical innovations that had radically altered the nature and role of geometry throughout mathematics and in the formulation of physical laws. Poincaré repeatedly asserts that we must release geometry from experiences that restrict our thinking—by contemplating how beings in a hyperbolic three-dimensional world would create their own geometry, and by recognising what mathematicians have already done by means of unrestricted geometric reasoning. Immediately after establishing a dictionary between the three-dimensional hyperbolic and Euclidean spaces, he states, 'But this is not all' and starts recounting the fruits of these striking shifts in geometric reasoning.[18]

'Consider what Klein and myself have done by using them in the integration of linear equations.' A decade earlier Poincaré had changed the essence of hyperbolic geometry from a mere curiosity of the geometer and an intriguing 'possibility' to a profound necessity for the analyst that lay hidden within much of mathematics. In 1880, examining issues regarding integrals of linear differential equations with algebraic coefficients, Poincaré had created the theory of automorphic functions, exploiting a series of deep and subtle insights all the way to the concept of the universal cover and to establishing and proving the uniformisation theorem that classified solutions of all analytic functions (as rational, elliptic or Fuchsian functions). As the historian of mathematics Jeremy Gray argues, Poincaré had devised a unique way of 'deriving' the essence of Riemann surfaces; he constructed them naturally from discontinuous groups, obtaining them as quotient spaces of the unit disc (rather than as branched coverings of the Riemann sphere), once he had the wonderful insight that the Möbius transformations he had used to define the class of Fuchsian functions were identical to the groups of motions of hyperbolic geometry. Poincaré's 'almost effortless introduction of Riemann surfaces' into his analysis was viewed as a 'dramatic novelty . . . especially since Riemann's ideas were still generally considered obscure and lacking in rigor'.[19]

A key move in his 1880 epochal advance occurred when Poincaré, while engaging research on indefinite ternary quadratic forms within number theory, conceptualised a new model of the hyperbolic plane, a hyperboloid model that projected to the open unit disc—establishing the existence of his Fuchsian groups and hyperbolic geometry at the heart of pure mathematics.[20] Poincaré's conceptualisation here is of a completely different nature from the establishment of Weierstrass coordinates, the Helmholtz hyperboloid model and other such models of much earlier date.[21] Poincaré details how to picture and construct this profound relationship in a self-analysis of his work (1884, 1886):

> One of the most important problems in the subject of indefinite ternary quadratic forms is the study of the discontinuous groups formed by the similarity substitutions, that is, linear substitutions which preserve the form. Let $F(x, y, z)$ be an indefinite quadratic form.
>
> We can choose the constant K so that $F(x, y, z) = K$ represents a hyperboloid of two sheets. The similarity substitutions then map a point on the hyperboloid to another point on the same sheet and, since the group is discontinuous, the hyperboloid becomes partitioned into infinitely many curvilinear polygons whose sides are diametric sections of the surface. A similarity substitution changes each polygon into another. We now take a perspective view by placing the eye at an umbilic of the surface and taking the plane of projection to be a

> circular section. One sheet of the hyperboloid is projected inside a circle, and the polygons drawn on this sheet project to curvilinear polygons bounded by circular arcs of the kind we have discussed in the theory of Fuchsian groups. Thus the study of similarity substitutions of quadratic forms reduces to that of Fuchsian groups, which is an unexpected rapprochement between two very different theories, and a new application of non-Euclidean geometry.[22]

Poincaré's surprising linkage of his hyperbolic disc and the two-sheeted hyperboloid created a new world of action within mathematics of profound power and import. The indefinite quadratic form producing the figure Poincaré instructs the reader to visualise (and construct) is equivalent to the indefinite quadratic form of flat two-dimensional spacetime; we will see that Minkowski carefully studied Poincaré's reasoning from these 1880 insights to Poincaré's 1905–6 creation of the four-dimensional metric (indefinite quadratic form) linking light, space and time.

Upon discovering Poincaré's theory of Fuchsian (automorphic) functions, Felix Klein initiated a correspondence with Poincaré the next day (12 June 1881), claiming in his second letter that Poincaré's 'analogy with non-Euclidean geometry does not hold' for more complicated cases without a limit circle. Poincaré immediately demonstrated that his analogy with non-Euclidean geometry *does* hold by creating his three-dimensional 'hyperbolic ball' model (the hyperbolic world), and fully generalising his theory of automorphic functions to all cases (naming cases without a limit circle Kleinian groups and functions).[23]

The litany continues. There is also a 'fourth geometry' with far more surprising properties than Riemann's or Lobachevsky's non-Euclidean geometries. Poincaré had established this fourth geometry and its astonishing properties in an 1887 article on the 'fundamental hypotheses of geometry', a geometry that implied three propositions 'so contrary to our habits of thought that the founders of geometry have denied them'.[24] In his 1891 litany, Poincaré mentions only one of its theorems and 'not the most surprising: a real line can be perpendicular to itself'.[25] This fourth geometry is the single-sheeted hyperboloid, which is precisely the two-dimensional hypersurface of de Sitter space, and one of its degeneracies is the geometry of two-dimensional Minkowski space.[26]

Poincaré continues to higher dimensions: from a group-theoretic theorem of Sophus Lie, we know there are a finite number of constant curvature *n*-dimensional geometries; but there are an infinite number of variable curvature Riemannian geometries, all depending on how the length of a curve is defined. The fundamental hypotheses of geometry take us much deeper and far beyond the need for a consistent and convenient metric geometry which can readily be translated into a different metric geometry if the need arises. At those depths lie the topological issues that Poincaré confronted in all of his mathematical creations—and that would shortly lead him to the topological link of the universal covering space and the fundamental group.

Poincaré then poses the following question: if several geometries are possible, which is true? He goes on to argue to his geometrical conventionalism and the puzzles regarding the hyperbolic world. He would reintroduce this 'hyperbolic world' in his 1895 article '*L'espace et la géométrie*',[27] coinciding with publication of his first innovations in 'Analysis Situs'. There, as in his 1903 fictions featuring the equivalence and hence 'indistinguishability' of the Hyperbolic and Euclidean worlds, Poincaré was getting at the notion of what space 'is'—not geometrically but topologically. No experiment can ever tell us which geometry (metric) is 'true' (natural) for our space, he argued, because we can only observe results of measurements involving objects—not of space itself nor of relations between objects and space. Poincaré established the 'amorphous continuum' as the underlying 'common basis' of the Euclidean and non-Euclidean metrics we can impose on the space.

As Poincaré emphasised in 1903, the 'worlds' so constructed are indistinguishable as different spaces 'if we can pass from one to the other by any point transformation whatever', adding that it is in this sense that it would be proper to understand the relativity of space.[28] Understanding the structure of our space, and our physics, requires mathematical constructions and the theorems of 'Analysis Situs' by which we delve beyond the impressions of experience.

4.4 THE 'TRUE RELATIONS' THAT PERSIST (A 'NATURAL KINSHIP') VERSUS THE ETHER (A 'GARMENT')

How did Poincaré engage the rationale, and impact the fate, of the notion of the ether in physics? This section seeks understanding through Poincaré's voice by tracing through his philosophical writings from 1901 to 1905—his 1901 updated self-analysis of his work and his first two collections of reprinted addresses and articles.

4.4.1 1886–1901: The 1900 Congresses, Poincaré's 1901 update, *Science and Hypothesis*

In 1901, Gösta Mittag-Leffler, the Swedish mathematician and editor of *Acta Mathematica*, asked Poincaré to prepare an update of the seventy-five-page self-analysis of his works that Poincaré had composed early in his career (1884, 1886) for induction into the Académie des Sciences. Poincaré's one-hundred-page update covers the fifteen-year period 1886–1901 and is organised into seven parts.[29] It was not published until 1921; Poincaré never wrote an update for 1901–12. Part 3 of the update, 'Diverse Questions of Pure Mathematics', explains how his need to create analysis situs deepened amid advances throughout his mathematical corpus. Part 4 of the update, 'Celestial Mechanics', features the concepts and methods (confronting analytic issues with topological and geometric reasoning) by which he revolutionised celestial mechanics and which became widely fruitful in astronomy and physics.

Part 5 of the update, 'Mathematical Physics', traces his contributions to the theory of partial differential equations and his critiques of physical theories, from early theoretical doubts about Fresnel's optical theory to Maxwell's electromagnetic theory of light and Lorentz's theory of the electron—loci of the physicists' ether. The long-standing concerns related to stellar aberration and Fresnel's dependence on the index of refraction, a shift in focus to time intervals involved in the transmission of light, and

Poincaré's 1898 considerations of the notion of simultaneity in astronomy and in measuring longitude—all this coalesced in Poincaré's 1900 metrological interpretation of 'local time' in Lorentz's theory.[30]

Part 6 of the update, 'Philosophy of Science', features, in a subsection entitled 'Physics', both Poincaré's 1897 ICM address and his 1900 ICP lecture assessing the status of the ether in mathematical, experimental and theoretical physics.[31] After stipulating that 'belief in the unity and simplicity of nature . . . is necessary for science', Poincaré insists on the need to distinguish between 'the foundation [*le fond*] and the form [*la forme*]' in physical theories—a terminology of his own invention that is key to how he views and contends with the physicists' ether within his mathematical physics:[32]

> The foundation is the existence of certain relationships between inaccessible objects. These relationships are the only reality we can achieve, and *all we can ask for is that there be the same relationships between these unknown real objects and the images we put in their place*.
>
> The form [image, analogy] is only a sort of garment with which we dress this skeleton [foundation, system of relationships]; we frequently change this garment, to the astonishment of the people of the world . . . But if the form changes often, the foundation remains.
>
> The hypotheses regarding what I have just called the form cannot be true or false, they can only be convenient or inconvenient. For example, the existence of the ether, [and] the existence of external objects, are only convenient hypotheses.[33]

For Poincaré, there is neither truth nor falsity regarding the existence of the ether; it is merely a convenient hypothesis, a garment freely chosen and discarded.

Most significant are two conclusions, and a warning, that Poincaré draws from the fact that the 'form' of the physical theory is merely a garment. First is his historical observation about the strength of science amid theory change: 'It is for this reason that certain theories which were believed to be permanently abandoned are reborn from their ashes.' Second is the conundrum of underdetermination: 'There are certain categories of facts which are equally well explained in two or more different theories, without any experience ever being able to decide between them.' Indeed, this underdetermination is 'particularly true for mechanistic theories', for 'it can be shown that if a phenomenon includes one mechanical explanation, it will have an infinity'.[34] Poincaré continues with a caution about the particularly dangerous allure of 'mechanism' in explaining physical phenomena:

> In any case, Mechanism is only one of the garments with which truth can be dressed, and if it satisfies our mind, we must not attach more importance to it than it deserves. It obliges us to introduce the hypothesis of auxiliary fluids such as ether; I present some views on the greater or lesser reality of this fluid.[35]

An infinite number of possible mechanical ether theories could be constructed and, indeed, a proliferating number appeared during the previous decade.

In contrast, for Poincaré, the ether is merely a term attached to relations that adhere in the physical phenomena of electromagnetism. His 1900 ICP address concludes: 'We must not forget that the goal of science is not Mechanism but unity [of explanation and understanding].' Poincaré refutes the so-called *bankruptcy of science* (the layman's view that the 'ephemeral nature of scientific theories' implies that 'they are absolutely in vain') with the example of Fresnel's theory of light as 'movements of the ether' in contrast to Maxwell's electromagnetic theory of light. The reality of the ether, and the particular form it took in the different theories, was not the relevant question: 'Fresnel's object was not to know whether there really is an ether . . . his object was to predict optical phenomena.' Fresnel's theory enables such prediction today 'as well as it did before Maxwell's time', not because of the reality of the theoretical objects or mechanical structure of Fresnel's theory, but because Fresnel's 'differential equations are always true'. These differential equations 'express relations, and if the equations remain true, it is because the relations preserve their reality'. The 'true relations' among these 'real objects which Nature will hide forever from our eyes', Poincaré insists, 'are the only reality we can attain'. What matters is that 'the same relations shall exist between these objects as between the images we are forced to put in their place'. We might find one image more convenient than another, but that choice does not change the underlying 'relations [that] are known to us' and that persist.[36]

Also in Part 6, in a subsection entitled 'Mechanics', Poincaré explains how concerns raised in his 1898 paper on the 'measure of time' reverberated throughout his 1900 ICPHIL critique of the 'principles of mechanics' (shortly reprinted in *Science and Hypothesis*), where he asks:

> May we not someday be compelled by new experiments to modify or even to abandon [the principles of mechanics]? These are questions that naturally arise, and the difficulty of solution is largely due to the fact that treatises on mechanics do not clearly distinguish between what is experiment, what is mathematical reasoning, what is convention, and what is hypothesis. That is not all.[37]

Poincaré argues that these distinct categories of practice enter into the conceptual difficulties within mechanics that he proceeds to itemise:

1. There is no absolute space, and we only conceive of relative motion; and yet in most cases mechanical facts are enunciated as if there were an absolute space to which they can be referred.
2. There is no absolute time. When we say that two periods are equal, the statement has no meaning, and can only acquire a meaning by convention.

It is a conceptually flawed and misleading practice, Poincaré warns, to enunciate mechanical facts with reference to an absolute space *as if it exists*. And, since there is no 'absolute time' (and since we cannot directly intuit equality for intervals), we can only establish a 'meaning' regarding the 'equality' of two time intervals *by convention*. Moreover, he cautions:

3. Not only have we no direct intuition of the equality of two periods, but we have not even direct intuition of the simultaneity of two events occurring in two different places. I have explained this in an article entitled '*Mesure du Temps*'.

Poincaré references his 1898 article explaining how the *problem of simultaneity of two distant events* is linked to the *problem of measuring time*, and how astronomers measure the velocity of light by supposing that it is constant and the same in all directions; thus, any statement about the simultaneity of spatially separated events is based on *a freely chosen convention*.[38] He concludes:

> 4. Finally, is not our Euclidean geometry in itself only a kind of convention of language? Mechanical facts might be enunciated with reference to a non-Euclidean space which would be less convenient but quite as legitimate as our ordinary space; the enunciation would become more complicated, but it still would be possible.[39]

Here is the challenge and the promise of Poincaré's geometric conventionalism—the invitation to consider an alternative non-Euclidean geometry for enunciating the facts of mechanics. Notwithstanding their experiential character, the principles of mechanics might just as legitimately (and perhaps more fruitfully) be 'enunciated' with reference to a 'space' that does not have the Euclidean metric.

Poincaré soon wrote the preface to *Science and Hypothesis*, there distinguishing variants of hypotheses for geometry, mechanics and the physical sciences:

> We therefore conclude that the *principles of geometry* are only conventions; but these conventions are not arbitrary, and if transported into another world (which I shall call the non-Euclidean world, and which I shall endeavour to describe), we shall find ourselves compelled to adopt more of them.
>
> In *mechanics* we shall be led to analogous conclusions, and we shall see that the principles of this science, although more directly based on experience, still share the conventional character of the geometrical postulates...
>
> But we now come to the *physical sciences*, properly so called, and here the scene changes. We meet with hypotheses of another kind, and we fully grasp how fruitful they are. No doubt at the outset theories seem unsound, and the history of science shows us how ephemeral they are; but they do not entirely perish, and of each of them some traces still remain. It is these traces which we must try to discover, because in them and in them alone is the true reality.[40]

The 'traces' that remain from now-discarded physical hypotheses—those relations which remain *despite theory change*—are the 'true relations of things' that are our sole access to 'true reality'.

Poincaré differentiates between types of hypotheses: (1) 'Some are *verifiable*, and when once confirmed by experiment become *truths of great fertility*', for empirically verifiable hypotheses propel science's advance. (2) Others 'may be useful to us in fixing our ideas' and guide our path to hypotheses of the first kind. (3) Still others 'are hypotheses only in appearance, and reduce to definitions or to conventions in disguise':

> The latter are to be met with especially in mathematics and in the sciences to which it is applied. From them, indeed, the sciences derive their rigor; such conventions are the result of the unrestricted activity of the mind, which in this domain recognizes no obstacle. For here the mind may affirm because it lays down its own laws; but let us clearly understand that while these laws are imposed on *our* science, which otherwise could not exist, they are not imposed on nature. Are they arbitrary? No; for if they were, they would not be fertile. Experience leaves us our freedom of choice, but it guides us by helping us to discern the most convenient path to follow.[41]

He distinguishes freely chosen conventions guided by experience from two antithetical extremes—conventionalism is *not* nominalism (mere names with no connection to physical reality), and conventionalism is *not* positivism (accessing 'things themselves'). Poincaré's argument is unambiguous: the aim of science is 'the relations between things; outside those relations there is no reality knowable' to science.[42]

4.4.2 1901–5: The crisis of theory, the 1904 St. Louis Congress, *The Value of Science*

Meanwhile, heated debates about the bankruptcy of science had fuelled distortions of Poincaré's 1900 ICP and ICPHIL lectures, issues he confronts in Part III, 'The Objective Value of Science', of his 1905 collection *La valeur de la science*. Chapters entitled 'Is Science Artificial' and 'Science and Reality' ask: 'Can science teach us the true relations of things?' Yes, Poincaré answers, distinguishing the *'form' taken by a physical theory* from the *underlying 'foundation' of 'true relations'* that remain true across different physical theories. While the form may change—as between Fresnel's and Maxwell's theories of light—the relations in the background foundation remain. Despite changing scientific hypotheses, the true relations 'will be found again under a new disguise in the other theories which will successively come to reign in place of the old'. Between 'the hypothetical currents which Maxwell supposes there are the same relations as between the hypothetical motions that Fresnel supposed'. Relations that remain true are our link to objective reality. So, 'what is the measure of their objectivity?'

> Well, it is precisely the same as for our belief in external objects . . . It may be said, for instance, that the ether is no less real than any external body; to say this body exists is to say there is between the color of this body, its taste, its smell, an intimate bond, solid and persistent; to say the ether exists is to say there is a *natural kinship between all the optical phenomena*, and neither of the two propositions has less value than the other . . . In sum, *the sole objective reality* consists in the relations of things whence results the universal harmony . . . [They] are objective because they are, will become, or will remain, common to all thinking beings.

The 'ether' for Poincaré is a name attached to the '*natural kinship* [*that exists*] *between all the optical phenomena*'—the objectively real relations that adhere in the phenomena and *persist* even with changes in the 'garment' of electromagnetic theory.[43]

Poincaré wrote these essays for *The Value of Science* to correct the 'strangest interpretations' (as if it were a defence of the Church against Galileo) of a comment Poincaré made at the 1900 ICPHIL regarding the truth status of propositions.[44] His address asserted that there is no absolute space and that we can only conceive of relative motion. In these essays, Poincaré explains the contested philosophical issue:

> No, there is no absolute space; these two contradictory propositions: 'The earth turns round' and 'The earth does not turn round' are, therefore, neither of them more true than the other. To affirm one while denying the other, *in the kinematic sense*, would be to admit the existence of absolute space.[45]

Poincaré relates the *kinematical impossibility* of determining absolute motion to the overriding question, can science teach us the true relations of things? At issue were profound challenges to our notions of space and time, high-stakes concerns he assessed in Part I, 'The Mathematical Sciences', in three reproduced articles: 'The Measure of Time' (1898), 'The Notion of Space' (1903), and 'Space and Its Three Dimensions' (1903)—directly linking 1898, when Poincaré asserted the metrological function of light and the conventionality of simultaneity, with 1903, when he completed his series 'Analysis Situs' regarding space of any dimension.

Poincaré's treatment of the crisis in theoretical physics putting these concepts at risk comprised Part II, 'The Physical Sciences', which links his 1897 ICM address 'Analysis and Physics' with his 1904 address to the St. Louis Congress (reproduced as three chapters assessing the history, present crisis and future of mathematical physics).[46] The St. Louis address encapsulates Poincaré's assessment of this crisis and contribution to its resolution during the intervening seven years, notably his statement of the principle of relativity and his oft-quoted concluding paragraph announcing the need for 'an entirely new mechanics' in which no apparent velocity can exceed that of light and in which observers in motion use a watch giving the 'local time'.[47]

A seminar analysing Poincaré's St. Louis address was held 31 January 1905, opening the Minkowski–Hilbert seminars on electron theory at Göttingen (held weekly until 31 July 1905); attendees of these seminars included Max Born, Max von Laue and Jakob Laub. While Poincaré's (5 June) and Einstein's (30 June) papers were not treated in this seminar, the seminar enabled the participants later to recognise the unique origin and substance of Einstein's purely kinematical theory of special relativity, in which ether had no role.[48] Our concern here is the unique substance of Poincaré's paper and how it established the mathematics of spacetime.

4.5 GÖTTINGEN, THE EXPANDING POINCARÉ CORPUS, AND MINKOWSKI'S PATH TO SPACETIME

This section discusses Minkowski's path to spacetime and sheds light on how Poincaré's contributions—his profound geometric and topological creations, his geometric conventionalism, his notions of space and time and his creation of spacetime geometry to establish 'true relations' that endure—became lost or distorted.

4.5.1 1890–1905: Pursuit of Poincaré's mathematics in the Klein–Hilbert–Minkowski seminars

For seventeen years (from 1890 to 1907), Klein ran a series of seminars (whose participants included Luigi Bianchi, William Osgood, Arthur Sommerfeld, Karl Schwarzschild and Hermann Weyl) that maintained a focused attention on Poincaré's works—reading and lecturing on them, dissecting and critiquing them and absorbing and incorporating their methods and results into the practices and sensibilities of the seminar participants. Poincaré's continuing creative flow kept Göttingen on high alert for his latest works.

David Hilbert's 1900 lecture to the second ICM identified twenty-three unsolved problems of great import, energising the goal to axiomatise and rigorise (thus render 'obsolete') what came before; but such work 'drew much of its strength from already flowing currents in 19th-century mathematical research'.[49] We see this strikingly in how Göttingen mathematicians intensively pursued and absorbed subtleties of Poincaré's reasoning.

In 1902, Poincaré returned a third time to his fourth geometry, highlighting its strange properties in his review of Hilbert's *Grundlagen der Geometrie*, prodding mathematicians to puzzle over why Poincaré finds it so important:

> Where would this new geometry rank in Hilbert's classification? We are glad to see that, as for the geometry of Riemann, all the axioms hold, except those of order and the axiom of Euclid; but whereas in the geometry of Riemann the axioms of order are false on all lines, in contrast, in the new geometry, *lines fall into two classes*, those on which the *axioms of order* are *true*, and those on which they are *false*.[50]

Poincaré would call upon these ('spacetime') relationships extensively in his 'Fifth Complement' to 'Analysis Situs'. Hilbert and his students and colleagues took note, as would Minkowski, for whom a third chair in mathematics was created at Göttingen in 1902. Hilbert and Minkowski held seminars featuring Poincaré's physics, focusing in 1903 on Poincaré's *New Methods of Celestial Mechanics* and many papers on 'figures of equilibrium of fluid masses', followed by mechanics (in 1904), and Lorentz's electron theory (in 1905, beginning with Poincaré's St. Louis lecture as noted in Section 4.4.2).[51]

In contemporaneous mathematics seminars, Minkowski lectured on Poincaré's 'Fifth Complement' (in late 1904), arithmetic and hyperbolic geometry (11 July 1905) and Fuchsian and Kleinian groups (25 July 1905). The claim that Minkowski's 11 July 1905 lecture presented a new model of hyperbolic geometry[52] is based on an article by Hans Jansen (from 1909)[53] that gave a detailed account of the hyperboloid model. But Jansen opens with a reference to Poincaré's (April 1881) Algiers paper[54] (where Poincaré derived the hyperboloid model within arithmetic in relation to his Fuchsian groups and functions), and then references Minkowski's lecture[55] as reporting arithmetic results he obtained based on the hyperboloid model. Minkowski's lecture on Poincaré's Fuchsian and Kleinian groups two weeks later would engage Poincaré's (June 1881) hyperbolic ball model. The focus on Poincaré's geometric moves was intensifying.

4.5.2 June 1905 through 1908: Poincaré's four-dimensional geometry, Minkowski's *Raum und Zeit*

Poincaré's 'On the Dynamics of the Electron' (a summary form) was read to the French academy on 5 June 1905; the paper appeared in January 1906.[56] As Darrigol explains, much in it was 'novel and important': a fully covariant formulation of 'Lorentz's transformations' (so named by Poincaré), 'its relativistic interpretation, its group-theoretic formulation, and its application to non-electromagnetic forces of cohesion or gravitation'.[57] Theoretical physicist Thibault Damour explains that most of Poincaré's 'key new results' are in the final section 'in a rather untransparent and unpedagogical form' but are mathematically complete: Poincaré had pioneered an 'elegant 4-dimensional geometrical formulation of Special Relativity' which Minkowski would expand upon in 1907–8 using more 'transparent notations'.[58]

We turn to Klein's opening lecture (1 November 1905) of the 1905–6 Klein–Hilbert–Minkowski mathematics seminar.[59] Stimulated by newly discovered notes from an 1859 course by Riemann on hypergeometric functions, the seminar would review related German works in hopes of attaining a proof (based on Riemannian principles) of Poincaré's 1883 uniformisation theorem (Hilbert's Problem #22). They would examine 'the work of Poincaré, beginning in 1881, published in the first five volumes of *Acta Mathematica*' and, in a few weeks, Klein would report 'to the [Mathematical Society] on the advances obtained by Poincaré and their relationship to my own investigations'.[60]

At the second session, Minkowski lectured again on Poincaré's 'Analysis Situs', featuring its results on multidimensional manifolds, and H. Mueller reported on Poincaré's work on Weierstrassian function theory. During the next five weeks, Gustav Herglotz gave lectures on Poincaré's uniformisation and Poincaré's '*Sur les résidus integrals doubles*' and Erhard Schmidt lectured on Poincaré's '*Sur les fonctions de deux variables*' while, at the Göttingen Mathematical Society, Ernst Zermelo lectured on Poincaré's work on boundary-value problems, and Schmidt on Poincaré's theory of differential equations—all this and more before the end of 1905.

At the eleventh session (31 January 1906), Klein lectured on Poincaré's development of Fuchsian functions, featuring Poincaré's stance on viewing the substitution group as a group of non-Euclidean rotations. Klein took the seminar participants through all of Poincaré's geometric moves, including a detailed account ('considered advantageous' by Minkowski) of Poincaré's description (quoted in Section 4.3) of how in 1880 he mapped from a calotte of one sheet of a two-sheeted hyperboloid to the unit disc.[61] The next five sessions continued reviewing Poincaré's theory of Fuchsian and Kleinian functions and groups, featuring Poincaré's general theory of automorphic forms that Klein (after Friedrich Schottky) had incorporated into his own programme with collaborators Ernst Ritter and Robert Fricke.[62]

On 13 June 1906, Poincaré submitted his proof of general uniformisation to *Acta Mathematica*.[63] Unaware of Poincaré's proof, Klein's seminars continued *almost a year* before Klein presented to the Göttingen Mathematical Society a proof of general uniformisation proposed by Paul Koebe (a seminar attendee). Koebe would provide a second proof on 19 November 1907 *that built upon (simplified, axiomatised) subtle novelties in Poincaré's proof* (which *Acta* had released earlier that month), novelties immediately absorbed into mainstream mathematical practice at Göttingen.

Klein's presentation of Koebe's first proof occurred on *the very same day* (11 May 1907) as Minkowski's first lecture on the equations of electrodynamics. Six months later (5 November 1907, *two weeks before Koebe's celebrated second proof*), Minkowski presented a second lecture, 'The Principle of Relativity',[64] in which he acknowledged much debt to Poincaré's January 1906 (Palermo) paper, highlighting Poincaré's four-dimensional treatment of gravitation. Minkowski's 1907 paper is evidence that 'it was Poincaré who most directly influenced the mathematics of Minkowski's space-time'.[65] And, as a founder of the Minkowski Institute for Foundational Studies states, it was 'Poincaré who first realized (before July 1905) that the Lorentz transformations have a natural geometric interpretation as rotations in a four-dimensional space whose fourth dimension is time'.[66]

Minkowski brought together Poincaré's 'true geometrical relations' of the four-dimensional space with the 'true physical relations' established by Einstein: as Minkowski wrote in his Cologne lecture '*Raum und Zeit*' on 21 September 1908, 'the credit of first recognising clearly that the time of one of the electrons is just as good as that of the other, that is to say, that t and t' are to be treated identically, belongs to A. Einstein'.[67] But Minkowski did not acknowledge Poincaré's crucial contribution in his Cologne lecture.[68] In the context of the Göttingen milieu, Minkowski's omission is not surprising, for both Klein and Hilbert, in different ways, had made it standard practice to absorb Poincaré's and others' contributions as elements of their own programmes, which they saw as superseding what came before.[69] The coincidence, in terms of both place and timing, of Koebe's two proofs of uniformisation with Minkowski's first two 1907 lectures on the principle of relativity may have played a role.

4.6 CONCLUDING COMMENTS: 1908–12 AND BEYOND

Minkowski's November 1907 lecture drawing upon Poincaré's Palermo paper remained unpublished until 1915. Poincaré never entered into priority disputes, but in 1912 he was asked to clarify his stance on the new mechanics. While many interpret Poincaré's 'Space and Time' paper as rejecting spacetime, mathematician Shlomo Sternberg argues that Poincaré's last four paragraphs establish his long-standing ownership of the geometry of the spacetime manifold. Poincaré died while this paper was in press, as was his paper 'Why Space Has Three Dimensions', where he developed a topological concept of dimension and emphasised how the amorphous continuum becomes space when a metric and other structures are imposed.[70] As these works unambiguously show, Poincaré had broken free from limitations of the physicist's ether concept, establishing 'true relations' that endure—a foundation for the electromagnetic and gravitational fields of relativistic and quantum phenomena.

Within a month after attending the 1911 first Solvay Conference as 'a newcomer to quantum ideas', Poincaré published three proofs at different levels of generality of the necessity of the quantum hypothesis. His proof made some prominent sceptics feel 'logically compelled to accept the quantum hypothesis in its entirety' and, in the 1920s, quantum physicists cemented Poincaré's newly invented 'integral-over-states' method at the heart of statistical mechanics, showing it to be 'completely justified'.[71]

Poincaré also wrote a recommendation for Einstein for a position at ETH Zürich, praising Einstein's unrivalled physical intuition and creativity:

> Einstein is one of the most original minds I have known ... [The] facility with which he has adapted to new conceptions and from which he knows how to draw the consequences ... translates immediately in his mind into the prediction of new phenomena, susceptible of being one day verified by experiment ... The future will show more and more the value of Mr. Einstein, and the university that finds a way to secure this young master is assured of drawing from it great honor.[72]

And, despite legends to the contrary, Einstein respected Poincaré's comments on quantum theory and dynamical systems at the conference, drawing upon them in his highly creative 1917 paper on quantum chaos.[73] Einstein was also well aware that many philosophers misinterpreted Poincaré's ideas, noting in 1919 that the German mathematician and philosopher Eduard Study (an early Klein protégé),[74]

> has treated him quite badly by pinning him down on a *truly superficial* comment about the practical significance of Euclidean geometry. Poincaré's expositions on the place of geometry within the whole scientific system appear to me to be considerably more profound.[75]

Einstein here picks out *precisely* the point which others—missing Poincaré's 'considerably more profound' mathematics and philosophy (and associated cosmological considerations)—distort into a dogma or premise.

Even physicists and mathematicians unaware of Poincaré's 'profound' contributions were immersed within the mathematical world he helped create, and, however unknowingly, used his tools, methods, reasoning and language. As the mathematician and physicist Hermann Weyl acknowledged in 1931:

> We differentiate now between the amorphous continuum and its metrical structures. The first has retained its *a priori* character ... whereas the structural field is completely subjected to the power-play of the world; being a real entity, Einstein prefers to call it the ether.[76]

The amorphous continuum, the mathematics of analysis situs, enables creation of the various 'structures' required in physics. Here we find Poincaré's legacy: the mathematical realm and methods of action needed to express physical relations within our space—however one choses to label them—relations so delicate, rich and precise.

NOTES

1. Henri Poincaré, 'Analysis and Physics', in *The Value of Science (VOS)*, trans. George Bruce Halsted (New York: The Science Press, 1907), 75–83, p. 76; orignally published as Henri Poincaré, '*Sur les rapports de l'analyse pure et de la physique mathématique*', *Acta Mathematica* 21 (1897): 331–41; republished (with the mathematical formulae removed) as Henri Poincaré, '*L'analyse et la physique*', in *La valeur de la science* (Paris: Flammarion, 1905).
2. Poincaré, 'Analysis and Physics', pp. 75–6.

3. Poincaré, 'Analysis and Physics', p. 76.
4. Albert Einstein, *Autobiographical Notes*, trans. and ed. Paul Arthur Schilpp (La Salle: Open Court, 1949), pp. 88–9. Einstein's final sentence reads: 'This exposition has fulfilled its purpose if it shows the reader how the efforts of a life hang together and why they have led to expectations of a certain kind.'
5. Henri Poincaré, '*Les géométries non euclidiennes*', *Revue générale des sciences pures et appliquées* 2 (1891): 769–74; trans. as 'Non-Euclidean Geometries' *Nature* 45 (1892): 404–7; reprinted as Chapters 3, 4 and 5 ('*Les géométries non euclidiennes*', '*L'espace et la géométrie*' and '*L'expérience et la géométrie*') in Part II: *L'Espace*' in Henri Poincaré, *La science et l'hypothèse* (Paris: Flammarion, 1902), pp. 49–67, 68–91, 92-109; this book was later translated into English as Henri Poincaré, *Science and Hypothesis* (*S&H*), trans. William Scott Greenstreet (London: Scott, 1905).
6. Henri Poincaré, '*Correspondance sur les géométries non euclidiennes* (*lettre à* M. Mouret),' ***Revue générale*** 3 (1892): 74–5. English translation by Peter Pesic, *Beyond Geometry: Classic Papers from Riemann to Einstein* (Mineola, NY: Dover, 2007): 105–7, p. 105.
7. Poincaré's conventionalism dominated much of twentieth-century philosophy of science, later joined by structural realist interpretations. See Michael Friedman, 'Einstein, Kant, and the Relativized *A Priori*', in Michel Bitbol, Pierre Kerszberg and Jean Petitot, eds., *Constituting Objectivity* (Dordrecht: Springer, 2009): 253–67; Yemima Ben-Menahem, *Conventionalism* (Cambridge: Cambridge University Press, 2006); 'Poincaré's Impact on Twentieth-Century Philosophy of Science', *HOPOS* 6 (2016): 257–73; Janet Folina, 'Poincaré and the Invention of Convention', in Maria de Paz and Robert DiSalle, eds., *Poincaré, Philosopher of Science* (Dordrecht: Springer, 2014), 25–45; Gerhard Heinzmann, 'Henri Poincaré and His Thoughts on the Philosophy of Science', in Éric Charpentier et al., eds., *The Scientific Legacy of Poincaré* (Providence: American Mathematical Society, 2010), 373–91, p. 374; Katherine Brading and Elise Crull, 'Epistemic Structural Realism and Poincaré's Philosophy of Science', *HOPOS* 7 (2017): 108–29.
8. Henri Poincaré, '*Analyse des travaux scientifiques de Henri Poincaré faite par lui-même*' (1901 update), *Acta Mathematica* 38 (1921; section printed 28 March 1913): 3–135, p. 101; my translation.
9. See Michael Friedman, 'Kant–*Naturphilosophie*–Electromagnetism', in Michael Friedman and Alfred Nordmann, eds., *The Kantian Legacy in Nineteenth-Century Science* (Cambridge: MIT Press, 2006), 51–79.
10. Several poets and artists (symbolists, cubists) seized upon Poincaré's mathematical conceptions: Linda Dalrymple Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art, Revised Edition* (Cambridge: MIT Press, 2013); Romano Nanni, '*Voir par l'intellect, voir par les yeux*', in Christina Vogel, ed., *Valéry et Léonard* (Frankfurt: Peter Lang, 2007), 141–52; Steven Cassedy, *Flight from Eden* (Berkeley: University of California Press, 1990).
11. Henri Poincaré, 'Analysis Situs', *Journal de l'Ecole Polytechnique* 1 (1895): 1–121; in Henri Poincaré, *Papers on Topology: Analysis Situs and Its Five Supplements*, trans. John Stillwell (Providence: American Mathematical Society, 2010), 5–74, p. 5; emphasis added.
12. Connemara Doran, *Seeking the Shape of the Universe: Confronting the Hyperbolic World, from Henri Poincaré to the Cosmic Microwave Background* (PhD dissertation, Harvard University, 2017).
13. Henri Poincaré, '*L'espace et ses trois dimensions*', *Revue de métaphysique et de morale* 11 (1903): 281–301, 407–29.
14. Poincaré, *VOS*, pp. 40–41; emphasis added.
15. Henri Poincaré, '*L'avenir des mathématiques*', Plenary Lecture at the ICM in Rome (April 1908). Translation by Jeremy Gray, 'Poincaré Replies to Hilbert: On the Future of Mathematics ca. 1908', *The Mathematical Intelligencer* 34 (2012): 15–29, p. 23.

16. Henri Poincaré,'Why Space Has Three Dimensions', *Mathematics and Science: Last Essays*, trans. John W. Bolduc (New York: Dover Publications, 1963): 25–44, p. 42; originally published as Henri Poincaré, *'Porquoi l'espace a trois dimensions'*, *Revue de métaphysique et de morale* 20 (1912): 483–504; *'Porquoi l'espace a trois dimensions'* in Henri Poincaré, *Dernières pensées* (Paris: Flammarion, 1913), 55–98.
17. Poincaré, *'Les géométries non euclidiennes'*.
18. Poincaré, *S&H*, p. 43.
19. Poincaré, *S&H*, p. 43; Jeremy Gray, 'The Three Supplements to Poincaré's Prize Essay of 1880 on Fuchsian Functions and Differential Equations', *Archives internationals d'histoire des sciences* 32 (1982): 221–35, p. 221. Henri Poincaré, *Three Supplements on Fuchsian Functions*, ed. Jeremy Gray and Scott Walter (Berlin: Akademie-Verlag, 1997). See assessments in Jeremy Gray, *Linear Differential Equations and Group Theory from Riemann to Poincaré* (Boston: Birkhäuser, 1986; second edition 2000); *Henri Poincaré: A Scientific Biography* (Princeton: Princeton University Press, 2013); '"The Soul of the Fact": Poincaré and Proof', *Studies in History and Philosophy of Modern Physics* 47 (2014): 142–50.
20. Poincaré reports this astonishing advance in the second supplement, submitted 6 September 1880 to the Académie des Sciences in Paris; and in one of two papers he presented at the Algiers meeting of the Académie des Sciences on 16 April 1881 (*Comptes rendus des sessions de l'Association Française pour l'Avancement des Sciences*, 10th Session, Algiers, 1881).
21. William Reynolds, 'Hyperbolic Geometry on a Hyperboloid', *The American Mathematical Monthly* 100, 5 (May 1993): 442–55, p. 453. Thomas Hawkins, 'Non-Euclidean geometry and Weierstrassian mathematics', *Historia Mathematica* 7 (1980): 289–342.
22. Henri Poincaré, *Sources of Hyperbolic Geometry*, trans. John Stillwell (Providence: American Mathematical Society, 1996), p. 121 (originally published as Henri Poincaré, *Notice sur les travaux scientifiques de M. Poincaré (rédigée par lui-même)* (Paris, Gauthier-Villars, 1884; second edition 1886), explaining his 1880–1 work); a figure by Konrad Polthier, Freie Universität Berlin, is reproduced at p. 120 and as the cover image.
23. Felix Klein to Henri Poincaré, 19 June 1881. Henri Poincaré to Felix Klein, 22 June 1881. '*Correspondance d'Henri Poincaré et de Felix Klein*', *Acta Mathematica* 39 (1923): 94–132. Poincaré delivered his generalisation to the Académie des Sciences in publications dated 27 June and 4 July 1881. Connemara Doran, 'Poincaré's Path to Uniformization', in Lizhen Ji and Shing-Tung Yau, eds., *Uniformization, Riemann–Hilbert Correspondence, Calabi–Yau Manifolds, and Picard–Fuchs Equations*, Advanced Lectures in Mathematics (Boston, International Press, 2018), 55–79.
24. Henri Poincaré, '*Sur les hypothèses fondamentales de la géométrie*', *Bulletin de la Société mathématique de France* 15 (1887): 203–16. English translation by Shlomo Sternberg, 'Review of *Imagery in Scientific Thought* by Arthur I. Miller', *The Mathematical Intelligencer* 8, 2 (1986): 65–74, p. 68. Poincaré's title, like Riemann's, featured 'the fundamental hypotheses' (not 'the facts') of geometry.
25. Poincaré, '*Les géométries non euclidiennes*', p. 772; my translation.
26. Sternberg, 'Review of *Imagery in Scientific Thought*', p. 68.
27. Henri Poincaré, '*L'espace et la géométrie*', *Revue de métaphysique et de morale* 3 (1895): 631–46.
28. Poincaré, *VOS*, p. 39; Poincaré, '*L'espace et ses trois dimensions*'.
29. Poincaré, '*Analyse des travaux scientifiques*', pp. 129–31; my translation.
30. John Stachel, 'Fresnel's (Dragging) Coefficient as a Challenge to 19th Century Optics of Moving Bodies', in Anne Kox and Jean Eisenstaedt, eds., *The Universe of General Relativity* (Boston: Birkhäuser, 2005): 1–13. Peter Galison, *Einstein's Clocks, Poincaré's Maps: Empires of*

Time (New York: Norton, 2003). Max Jammer, *Concepts of Simultaneity* (Baltimore: Johns Hopkins University Press, 2006). Olivier Darrigol, 'Poincaré's Light', in *Poincaré, 1912–2012*, ed. Bertrand Duplantier and Vincent Rivasseau (Basel: Springer Birkhäuser, 2012): 1–50.

31. The update cites Henri Poincaré, '*Préface*', in *Théorie mathématique de la lumière* (Paris: Publications chez G. Carré et Naud, 1889), pp. i–iv, and the 1900 ICP address '*Sur les rapports de la physique mathématique et de la physique expérimentale*', which was immediately published as Henri Poincaré, '*Les relations entre la physique expérimentale et la physique mathématique*', in C.-E. Guillaume and L. Poincaré, eds., *Rapports présentés au Congrès international de physique réuni à Paris en 1900 sous les auspices de la Société française de physique, rassemblés et publiés*, vol. 1 (Paris: Gauthier-Villars, 1900), 1–29, and Henri Poincaré, '*Les relations entre la physique expérimentale et la physique mathématique*', *Revue générale des sciences pures et appliquées* 11 (1900): 1163–75; it was then republished in 1902 as Henri Poincaré, '*Les hypothèses en physique*', in Poincaré, *La science et l'hypothèse*, 167–88, and Poincaré, '*Les théories de la physique moderne*', in Poincaré, *La science et l'hypothèse*, 189–212.
32. Poincaré, *'Analyse des travaux scientifiques'*, p. 130.
33. Poincaré, *'Analyse des travaux scientifiques'*, p. 130; emphasis added.
34. Poincaré, *'Analyse des travaux scientifiques'*, p. 130.
35. Poincaré, *'Analyse des travaux scientifiques'*, p. 131.
36. Poincaré, *S&H*, pp. 160–1.
37. Henri Poincaré, '*Sur les principes de la mécanique*', *Bibliothèque du Congrès international de philosophie*, vol. 3 (Paris, 1901): 457–94; p. 457; Poincaré, 'The Classical Mechanics', in *S&H*, 89–110, p. 89, and Poincaré, 'Relative and Absolute Motion', in *S&H*, pp. 111–22.
38. Henri Poincaré, '*La mesure du temps*', *Revue de métaphysique et de morale* 6 (1898): 1–13; Henri Poincaré, The 'Measure of Time', in Poincaré, *VOS*, 26–36.
39. Poincaré, Chapter 6 'The Classical Mechanics', *S&H*, p. 90.
40. Poincaré, *S&H*, pp. xxv–xxvi.
41. Poincaré, *S&H*, pp. xxii–xxiii.
42. Poincaré, *S&H*, p. xxiv.
43. Poincaré, *VOS*, pp. 138–40; emphasis added.
44. Poincaré, *VOS*, p. 140.
45. Poincaré, *VOS*, p. 141; Poincaré's emphasis.
46. Henri Poincaré, '*L'état actuel et l'avenir de la physique mathématique*', *Bulletin des sciences mathématiques* 28, 2nd series (1904): 302–24; trans. George Bruce Halsted, *The Monist* 15 (1905): 1–24.
47. Poincaré, *VOS*, p. 111.
48. Richard Staley, *Einstein's Generation: The Origins of the Relativity Revolution* (Chicago: University of Chicago Press, 2008), pp. 369–75.
49. David Rowe, 'Klein, Hilbert, and the Göttingen Mathematical Tradition', *Osiris*, 2nd Series, 5 (1989): 186–213, p. 199; 'The Calm before the Storm', in Vincent Hendricks, Stig Andur Pedersen and Klaus Frovin Jørgensen, *Proof Theory* (Dordrecht: Springer, 2000), p. 71.
50. Henri Poincaré, '*Les fondements de la géométrie*: *Grundlagen der Geometrie par* M. Hilbert, *professeur à l'Université de Göttingen*', *Bulletin des sciences mathématiques* 26 (1902): 249–72, reprinted in Henri Poincaré, *Scientific Opportunism, L'Opportunisme scientifique: An Anthology*, comp. Louis Rougier, ed. Laurent Rollet (Basel: Birkhäuser, 2002), 33-46, p. 44; my translation; emphasis added.
51. Scott Walter, 'Breaking in the 4-Vectors', in Jürgen Renn and Matthias Schemmel, eds., *The Genesis of General Relativity*, vol. 3 (Berlin: Springer, 2007): 193–252, p. 214; 'Minkowski, Mathematicians, and the Mathematical Theory of Relativity', in Hubert Goenner, Jürgen Renn, Jim Ritter and Tilman Sauer, eds., *The Expanding Worlds of General Relativity* (Boston:

Birkhäuser, 1999): 45–86; 'Minkowski's Modern World', in Vesselin Petkov, ed., *Minkowski Spacetime: A Hundred Years Later* (Dordrecht: Springer, 2010), 43–61. Leo Corry, 'Hermann Minkowski, Relativity and the Axiomatic Approach to Physics', in Petkov, *Minkowski Spacetime*, 3–41; Peter Galison, 'Minkowski's Space-Time: From Visual Thinking to the Absolute World', *Historical Studies in the Physical Sciences* 10 (1979): 85–121.

52. Apparently misreading Reynolds, 'Hyperbolic Geometry on a Hyperboloid', which correctly situates Poincaré's 1880–1 papers (see notes 19 and 20), Poincaré's 1887 paper (Poincaré, '*Sur les hypothèses fondamentales de la géométrie*'), Minkowski's 1907 article (published posthumously as Hermann Minkowski, '*Das Relativitätsprinzip*', *Annalen der Physik* 352 (1915): 927–38) and Jansen's 1909 article, published as Hans Jansen, '*Abbildung der hyperbolischen Geometrie auf ein zweischaliges Hyperboloid*', *Mitteilungen der Mathematische Gesellschaft in Hamburg* 4 (1909): 409–40 (not noticing Poincaré's 1905–6 articles (see note 56)).
53. Jansen, '*Abbildung der hyperbolischen Geometrie auf ein zweischaliges Hyperboloid*', p. 409.
54. Jansen references H. Poincaré, '*Sur les applications de la géométrie non euclidienne à la théorie des formes quadratiques*', in *Association française pour l'avancement des sciences, Compte Rendu de la 10*E *Session. Alger. 1881* (Paris: *Association française pour l'avancement des sciences*, 1881): 132–8.
55. Jansen references: Hermann Minkowski, '*Jahresbericht der Deutschen Mathematiker-Vereinigung*' ed. by August Gutzmer (Leipzig: Teubner, 1905).
56. Henri Poincaré, '*Sur la dynamique de l'électron*', *Comptes rendus* 140 (1905): 1504–8, and '*Sur la dynamique de l'électron*', *Rendiconti del circolo matematico di Palermo* 21 (1906): 129–76.
57. Darrigol, 'Poincaré's Light', p. 39.
58. Thibault Damour, 'What Is Missing from Minkowski's "*Raum und Zeit*" lecture', *Annalen der Physik* 17 (2008): 619–30, p. 625.
59. Felix Klein, 'Mathematical Seminar at Göttingen, Winter Semester 1905/1906 under the Direction of Professors Klein, Hilbert, Minkowski. Talks by F. Klein. Notes by Dr. Phil. Otto Toeplitz', http://www.claymath.org/sites/default/files/klein1math.sem_._ws1905-06.pdf, accessed 29 March 2018.
60. Klein, 'Mathematical Seminar', p. 3.
61. Felix Klein, '*Elfte Sitzung*', in *Klein Protokolle, Band* 23, *Seite* 85, http://page.mi.fu-berlin.de/moritz/klein/#id-988, accessed 29 March 2018.
62. Umberto Bottazzini and Jeremy Gray, *Hidden Harmony: Geometric Fantasies: The Rise of Complex Function Theory* (New York: Springer, 2013), p. 611.
63. Doran, 'Poincaré's Path to Uniformization'.
64. Minkowski, '*Das Relativitätsprinzip*'.
65. Galison, 'Minkowski space-time', 94–5, noting that Minkowski and Poincaré differed in their ontological commitments.
66. Petkov, 'Preface', *Minkowski Spacetime*, p. vi.
67. Hermann Minkowski, '*Raum und Zeit*', *Jahresberichte der Deutschen Mathematiker-Vereinigung* (Leipzig: Teubner, 1909); 'Space and Time', trans. Dennis Lehmkuhl in Petkov, *Minkowski Spacetime*, p. xxv.
68. See Damour, 'What Is Missing'.
69. Doran, 'Seeking the Shape of the Universe'.
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